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Magnetic Curves According to Bishop Frame and Type-2 Bishop Frame in Euclidean 3-Space

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Authors' contributions

This work was carried out in collaboration between both authors. Authors AK and HBK designed the study, obtained the main definitions and results, wrote the first draft of the manuscript and managed literature searches together. Both authors read and approved the final manuscript.

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Abstract

In this paper, we define the notions of T-magnetic, N_1-magnetic, N_2-magnetic curves according to Bishop frame and ξ_1-magnetic, ξ_2-magnetic, B-magnetic curves according to type-2 Bishop frame in Euclidean 3-space. Also, we obtain the magnetic vector field V when the curve is T-magnetic, N_1-magnetic, N_2-magnetic trajectory of V according to Bishop frame and ξ_1-magnetic, ξ_2-magnetic, B-magnetic trajectory of V according to type-2 Bishop frame. Finally, we give an example for magnetic curves according to Bishop frame and type-2 Bishop frame.

Keywords: Magnetic curves; lorentz force; frenet frame; bishop frame; type-2 bishop frame.

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1 Introduction

The magnetic curves on a Riemannian manifold (M, g) are trajectories of charged particles moving on M under the action of a magnetic field F. A *magnetic field* is a closed 2-form F on M and the *Lorentz force* of the magnetic field F on (M, g) is a (1,1)-tensor field Φ given by $g(\Phi(X), Y) = F(X, Y)$, for any vector fields $X, Y \in \chi(M)$. In dimension 3, the magnetic fields may be defined using divergence-free vector fields. As Killing vector fields have zero divergence, one may define a special class of magnetic fields called *Killing magnetic fields*.

Different approaches in the study of magnetic curves for a certain magnetic field and on the fixed energy level have been rewieved by Munteanu in [1]. He has emphasized them in the case when the magnetic trajectory corresponds to a Killing vector field associated to a screw motion in the Euclidean 3-space. In [2], the authors have investigated the trajectories of charged particles moving in a space modeled by the homogeneous 3-space $S^2 \times \mathbb{R}$ under the action of the Killing magnetic fields.

In [3], the authors have classified all magnetic curves in the 3-dimensional Minkowski space corresponding to the Killing magnetic field $V = a\partial_x + b\partial_y + c\partial_z$, with $a, b, c \in \mathbb{R}$. They have found that, they are helices in E_1^3 and draw the most relevant of them. In 3D semi-Riemannian manifolds, Özdemir et al. have determined the notions of T-magnetic, N-magnetic and B-magnetic curves and give some characterizations for them [4]. Also, in [5], the authors have studied on magnetic pseudo null and magnetic null curves in Minkowski 3-space.

In any 3D Riemannian manifold (M, g), magnetic fields of nonzero constant length are one to one correspondence to almost contact structure compatible to the metric g. From this fact, many authors have motivated to study magnetic curves with closed fundamental 2-form in almost contact metric 3-manifolds, Sasakian manifolds, quasi-para-Sasakian manifolds and etc (see [6], [7], [8], [9]).

On the other hand, the local theory of space curves has been studied by many mathematicians by using Frenet-Serret theorem. The Frenet frame is constructed for the curve of 3-time continuously differentiable non-degenerate curves. But, if the second derivative of the curve is zero, then the curvature may vanish at some points on the curve. For this reason, we need an alternative frame in E^3. Hence, an alternative moving frame along a curve is defined by Bishop in 1975 and he called it Bishop frame or parallel transport frame which is well defined as well the curve has vanishing second curvature [10]. The Bishop frame have many applications in Biology and Computer Graphics. For example it may be possible to compute information about the shape of sequences of DNA using a curve defined by the Bishop frame. The Bishop frame may also provide a new way to control virtual cameras in computer animations [11]. After defining this useful alternative frame, many studies have been done by mathematicians using it and type-2 Bishop frame in E^3 Euclidean space and E_1^3 Minkowski space (see [11], [12], [13], [14], [15] and etc).

In this study, the third chapter is devoted to the notions of T-magnetic, N_1-magnetic and N_2-magnetic curves according to Bishop frame in Euclidean 3-space. In the third chapter, also we obtain the magnetic vector field V when the curve is a T-magnetic, N_1-magnetic and N_2-magnetic trajectory of V according to Bishop frame. In the fourth chapter, we investigate the ξ_1-magnetic, ξ_2-magnetic and B-magnetic curves according to type-2 Bishop frame in Euclidean 3-space and we obtain the magnetic vector field V when the curve is a ξ_1-magnetic, ξ_2-magnetic and B-magnetic trajectory of V according to type-2 Bishop frame. Finally we give an example for magnetic curves according to Bishop frame and type-2 Bishop frame.

2 Preliminaries

Firstly, we will recall some fundamental notions about curves in Euclidean 3-space E^3.

If T, N and B are unit tangent vector field, unit principal normal vector field and unit binormal vector field of a space curve α, respectively, then $\{T, N, B\}$ is called the moving *Frenet frame* of α and the Frenet-Serret formulae is given by

$$\begin{bmatrix} T' \\ N' \\ B' \end{bmatrix} = \begin{bmatrix} 0 & \kappa & 0 \\ -\kappa & 0 & \tau \\ 0 & -\tau & 0 \end{bmatrix} \begin{bmatrix} T \\ N \\ B \end{bmatrix}, \tag{2.1}$$

where

$$g(T, T) = g(N, N) = g(B, B) = 1, \; g(T, N) = g(N, B) = g(B, T) = 0. \tag{2.2}$$

Here κ and τ are curvature functions of the curve α [16].

An alternative moving frame along a curve is defined by Bishop in 1975 [10]. For defining an alternative moving frame which is called *Bishop frame* or *parallel transport frame* in E^3, one can parallel transport each component of an orthonormal frame along the curve. Moreover, this frame is well defined as well the curve has vanishing second curvature. The Bishop frame is written as

$$\begin{bmatrix} T' \\ N_1' \\ N_2' \end{bmatrix} = \begin{bmatrix} 0 & k_1 & k_2 \\ -k_1 & 0 & 0 \\ -k_2 & 0 & 0 \end{bmatrix} \begin{bmatrix} T \\ N_1 \\ N_2 \end{bmatrix}, \tag{2.3}$$

where T is the tangent vector of the curve and $\{N_1, N_2\}$ are any convenient arbitrary basis for the remainder of the frame. Here, $\{T, N_1, N_2\}$ is called *Bishop trihedra* and k_1 and k_2 are called *Bishop curvatures* of the curve α. The relation between Frenet frame and Bishop frame is given by

$$\begin{bmatrix} T \\ N \\ B \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta(t) & \sin\theta(t) \\ 0 & -\sin\theta(t) & \cos\theta(t) \end{bmatrix} \begin{bmatrix} T \\ N_1 \\ N_2 \end{bmatrix}, \tag{2.4}$$

where $\theta(t) = \arctan \frac{k_2}{k_1}$, $\tau(t) = \theta'(t)$ and $\kappa(t) = \sqrt{(k_1)^2 + (k_2)^2}$. The Bishop curvatures are defined by $k_1 = \kappa \cos\theta(t)$, $k_2 = \kappa \sin\theta(t)$.

Another relatively parallel adapted frame is called *type-2 Bishop frame* and it is defined by

$$\begin{bmatrix} \xi_1' \\ \xi_2' \\ B' \end{bmatrix} = \begin{bmatrix} 0 & 0 & -\varepsilon_1 \\ 0 & 0 & -\varepsilon_2 \\ \varepsilon_1 & \varepsilon_2 & 0 \end{bmatrix} \begin{bmatrix} \xi_1 \\ \xi_2 \\ B \end{bmatrix} \tag{2.5}$$

and the relation between Frenet frame and type-2 Bishop frame is given by

$$\begin{bmatrix} T \\ N \\ B \end{bmatrix} = \begin{bmatrix} \sin\theta(t) & -\cos\theta(t) & 0 \\ \cos\theta(t) & \sin\theta(t) & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \xi_1 \\ \xi_2 \\ B \end{bmatrix}, \tag{2.6}$$

where $\theta(t) = \arctan \frac{\varepsilon_2}{\varepsilon_1}$, $\kappa(t) = \theta'(t)$ and $\tau = \sqrt{(\varepsilon_1)^2 + (\varepsilon_2)^2}$. Here, $\{\xi_1, \xi_2, B, \varepsilon_1, \varepsilon_2\}$ is called the *type-2 Bishop apparatus* of the curve $\alpha = \alpha(t)$ and the *type-2 Bishop curvatures* are defined by $\varepsilon_1(t) = -\tau \cos\theta(t)$, $\varepsilon_2(t) = -\tau \sin\theta(t)$ (for detail see [10], [12], [15] and etc.).

Now, we will give some informations about the magnetic curves in 3-dimensional semi-Riemannian manifolds.

A divergence-free vector field defines a magnetic field in a three-dimensional semi-Riemannian manifold M. It is known that, $V \in \chi(M^n)$ is a Killing vector field if and only if $L_V g = 0$ or, equivalently, $\nabla V(p)$ is a skew-symmetric operator in $T_p(M^n)$, at each point $p \in M^n$. It is clear that, any Killing vector field on (M^n, g) is divergence-free. In particular, if $n = 3$, then every Killing vector field defines a magnetic field that will be called a *Killing magnetic field* [17].

Let (M, g) be an n-dimensional semi-Riemannian manifold. A *magnetic field* is a closed 2-form F on M and the *Lorentz force* Φ of the magnetic field F on (M, g) is defined to be a skew-symmetric operator given by

$$g(\Phi(X), Y) = F(X, Y), \quad \forall X, Y \in \chi(M). \tag{2.7}$$

The *magnetic trajectories* of F are curves α on M that satisfy the *Lorentz equation* (sometimes called the *Newton equation*)

$$\nabla_{\alpha'}\alpha' = \Phi(\alpha'). \tag{2.8}$$

The Lorentz equation generalizes the equation satisfied by the geodesics of M, namely $\nabla_{\alpha'}\alpha' = 0$.

Note that, one can define on M the cross product of two vectors $X, Y \in \chi(M)$ as follows

$$g(X \times Y, Z) = dv_g(X, Y, Z), \quad \forall Z \in \chi(M).$$

If V is a Killing vector field on M, let $F_V = \imath_V dv_g$ be the corresponding Killing magnetic field. By $\imath$ we denote the inner product. Then, the Lorentz force of F_V is

$$\Phi(X) = V \times X.$$

Consequently, the Lorentz force equation (2.8) can be written as

$$\nabla_{\alpha'}\alpha' = V \times \alpha' \tag{2.9}$$

(for detail see [1], [4]).

Let $\alpha(t)$ be a non-null immersed curve in a 3-dimensional Lorentzian space form M with sectional curvature c and let $v(t) = |\alpha'(t)|$ be the speed of α. Let us consider a variation of α, $\Gamma = \Gamma(t, z)$: $I \times (-\varepsilon, \varepsilon) \longrightarrow M$ with $\Gamma(t, 0) = \alpha(t)$. In particular, one can choose $\varepsilon > 0$ in such way that t-curves of the variation have the same causal character as that of α. Associated with Γ there are two vector fields along Γ, $V(t, z) = \frac{\partial \Gamma}{\partial z}(t, z)$ and $W(t, z) = \frac{\partial \Gamma}{\partial t}(t, z)$. In particular, $V(t) = V(t, 0)$ is the variational vector field along α and $W(t, z)$ is the tangent vector field of the t-curves. We will use the notation $V = V(t, z)$, $v = v(t, z)$, $\kappa = \kappa(t, z)$, etc. with the obvious meanings. Also, if s denotes the arclength parameter of the t-curves, we will write $V(s, z)$, $v(s, z)$, $\kappa(s, z)$, etc. for the corresponding reparametrizations.

So we can give the following Lemma [18]:

Lemma 2.1. *With the above notation, the following assertions hold:*

i) $[V, W] = 0$;

ii) $V(v) = \frac{\partial v}{\partial z}(t, 0) = -\varepsilon_1 g(\nabla_T V, T)v$;

iii) $V(\kappa) = \frac{\partial \kappa^2}{\partial z}(t, 0) = 2\varepsilon_2 g(\nabla_T^2 V, \nabla_T T) + 4\varepsilon_1 \kappa^2 g(\nabla_T V, T) + 2\varepsilon_2 g(R(V, T)T, \nabla_T T)$;

iv) $V(\tau) = \frac{\partial \tau^2}{\partial z}(t, 0) = -2\varepsilon_2 g(\frac{1}{\kappa}\nabla_T^3 V - \frac{\kappa'}{\kappa^2}\nabla_T^2 V + \varepsilon_1(\varepsilon_2\kappa + \frac{c}{\kappa})\nabla_T V - \varepsilon_1 \frac{c\kappa'}{\kappa^2} V, \tau B)$,

where $\varepsilon_1 = g(T, T), \varepsilon_2 = g(N, N)$, $\varepsilon_3 = g(B, B)$, *R and c are curvature tensor and sectional curvature of M, respectively and* $\kappa' = \frac{\partial \kappa}{\partial t}(t, 0)$.

If $V(t)$ is the restriction to $\alpha(t)$ of a Killing vector field, then the vector field V satisfies the following condition (for detail, see [18]):

$$V(v) = V(\kappa) = V(\tau) = 0. \tag{2.10}$$

3 Magnetic Curves According To Bishop Frame in Euclidean 3-Space

In this section, we investigate the T-magnetic, N_1-magnetic and N_2-magnetic curves according to Bishop frame in Euclidean 3-space. Also, we obtain the Killing magnetic vector field V when the curve is a T-magnetic, N_1-magnetic and N_2-magnetic trajectory of V according to Bishop frame.

3.1 T-Magnetic curves according to Bishop frame in Euclidean 3-space

Definition 3.1. Let $\alpha : I \subset \mathbb{R} \longrightarrow \mathbb{R}^3$ be a curve with Bishop frame in Euclidean 3-space and F_V be a magnetic field in $\mathbb{R}^3$. If the tangent vector field T of the Bishop frame satisfies the Lorentz force equation $\nabla_{\alpha'} T = \Phi(T) = V \times T$, then the curve α is called a **T-magnetic curve according to Bishop frame**.

Proposition 3.1. *Let α be a unit speed T-magnetic curve according to Bishop frame in Euclidean 3-space. Then, the Lorentz force according to the Bishop frame is obtained as*

$$\begin{bmatrix} \Phi(T) \\ \Phi(N_1) \\ \Phi(N_2) \end{bmatrix} = \begin{bmatrix} 0 & k_1 & k_2 \\ -k_1 & 0 & \rho \\ -k_2 & -\rho & 0 \end{bmatrix} \begin{bmatrix} T \\ N_1 \\ N_2 \end{bmatrix}, \tag{3.1}$$

where ρ is a certain function defined by $\rho = g(\Phi N_1, N_2)$.

Proof. Let α be a T-magnetic curve according to Bishop frame in Euclidean 3-space with the Bishop apparatus $\{T, N_1, N_2, k_1, k_2\}$. From the definition of the T-magnetic curve according to Bishop frame, we know that $\Phi(T) = k_1 N_1 + k_2 N_2$. On the other hand, since $\Phi(N_1) \in Sp\{T, N_1, N_2\}$, we have $\Phi(N_1) = a_1 T + a_2 N_1 + a_3 N_2$. So, we get

$$\begin{aligned} a_1 &= g(\Phi N_1, T) = -g(N_1, \Phi T) = -g(N_1, k_1 N_1 + k_2 N_2) = -k_1, \\ a_2 &= g(\Phi N_1, N_1) = 0, \\ a_3 &= g(\Phi N_1, N_2) = \rho \end{aligned}$$

and hence we obtain that, $\Phi(N_1) = -k_1 T + \rho N_2$.

Furthermore, from $\Phi(N_2) = b_1 T + b_2 N_1 + b_3 N_2$, we have

$$\begin{aligned} b_1 &= g(\Phi N_2, T) = -g(N_2, \Phi T) = -g(N_2, k_1 N_1 + k_2 N_2) = -k_2, \\ b_2 &= g(\Phi N_2, N_1) = -g(N_2, \Phi N_1) = -\rho, \\ b_3 &= g(\Phi N_2, N_2) = 0 \end{aligned}$$

and so, we can write $\Phi(N_2) = -k_2 T - \rho N_1$, which completes the proof. □

Proposition 3.2. *Let α be a unit speed T-magnetic curve according to Bishop frame in Euclidean 3-space. Then, the curve α is T-magnetic trajectory of Killing magnetic vector field V if and only if the Killing magnetic vector field V is*

$$V = \rho T - k_2 N_1 + k_1 N_2 \tag{3.2}$$

along the curve α.

Proof. Let α be T-magnetic trajectory of Killing magnetic vector field V according to Bishop frame. Using Proposition 3.1 and taking $V = aT + bN_1 + cN_2$; from $\Phi(T) = V \times T$, we get

$$b = -k_2, \; c = k_1;$$

from $\Phi(N_1) = V \times N_1$, we get

$$a = \rho, \; c = k_1$$

and from $\Phi(N_2) = V \times N_2$, we get

$$a = \rho, \; b = -k_2$$

and so the Killing magnetic vector field V can be written by (3.2). Conversely, if the Killing magnetic vector field V is the form of (3.2), then one can easily see that $V \times T = \Phi(T)$ holds. So, the curve α is T-magnetic projectory of the Killing magnetic vector field V according to Bishop frame. □

Theorem 3.1. *Let α be a unit speed T-magnetic curve according to Bishop frame in a 3-dimensional Euclidean space form M with sectional curvature c and V be a Killing vector field on a simply connected space form $(M(c), g)$. If the curve α is one of the T-magnetic trajectories of $(M(c), g, V)$, then we have the following statements:*

i) $\rho =$constant,

ii) the harmonic curvature function of the curve α according to Bishop frame is

$$H = \frac{\rho k_2' + k_1''}{k_2'' - \rho k_1'} \tag{3.3}$$

or

$$H = \frac{-k_1 k_1' k_2 + \rho k_1'' - k_2''' - {k_2}^2 k_2' + c\rho k_1 - ck_2' - \frac{k_1 k_1' + k_2 k_2'}{{k_1}^2 + {k_2}^2}(\rho k_1' - k_2'' - ck_2)}{k_1 k_2 k_2' + \rho k_2'' + k_1''' + {k_1}^2 k_1' + c\rho k_2 + ck_1' - \frac{k_1 k_1' + k_2 k_2'}{{k_1}^2 + {k_2}^2}(\rho k_2' + k_1'' + ck_1)}. \tag{3.4}$$

Proof. If V is a T-magnetic vector field in a 3-dimensional Euclidean space form M, then it is the form of (3.2). Differentiating (3.2) with respect to s, from (2.3) we get

$$\nabla_T V = \rho' T + \{\rho k_1 - k_2'\} N_1 + \{\rho k_2 + k_1'\} N_2. \tag{3.5}$$

Using $V(v) = 0$ in Lemma 2.1, we have

$$\rho = constant. \tag{3.6}$$

Differentiating (3.5) with respect to s and using (3.6), from (2.3) we get

$$\nabla_T^2 V = \{-\rho {k_1}^2 + k_1 k_2' - \rho {k_2}^2 - k_1' k_2\} T + \{\rho k_1' - k_2''\} N_1 + \{\rho k_2' + k_1''\} N_2 \tag{3.7}$$

and differentiating (3.7) with respect to s once more, from (2.3) we get

$$\begin{aligned} \nabla_T^3 V \;\; = \;\; & \{-3\rho k_1 k_1' + 2k_1 k_2'' - 3\rho k_2 k_2' - 2k_1'' k_2\} T \\ & + \{-\rho {k_1}^3 + {k_1}^2 k_2' - \rho k_1 {k_2}^2 - k_1 k_1' k_2 + \rho k_1'' - k_2'''\} N_1 \\ & + \{-\rho {k_2}^3 - k_1' {k_2}^2 - \rho {k_1}^2 k_2 + k_1 k_2 k_2' + \rho k_2'' + k_1'''\} N_2. \end{aligned} \tag{3.8}$$

Since $V(\kappa) = 0$ and $V(\tau) = 0$ in Lemma 2.1, using (3.5)-(3.8) and the definition of the Bishop frame, we reach to (3.3) and (3.4), respectively. □

3.2 N_1-Magnetic curves according to Bishop frame in Euclidean 3-space

Definition 3.2. Let $\alpha : I \subset \mathbb{R} \longrightarrow \mathbb{R}^3$ be a curve with Bishop frame in Euclidean 3-space and F_V be a magnetic field in $\mathbb{R}^3$. If the vector field N_1 of the Bishop frame satisfies the Lorentz force equation $\nabla_{\alpha'} N_1 = \Phi(N_1) = V \times N_1$, then the curve α is called an N_1-**magnetic curve according to Bishop frame**.

Proposition 3.3. *Let α be a unit speed N_1-magnetic curve according to Bishop frame in Euclidean 3-space. Then, the Lorentz force according to the Bishop frame is obtained as*

$$\begin{bmatrix} \Phi(T) \\ \Phi(N_1) \\ \Phi(N_2) \end{bmatrix} = \begin{bmatrix} 0 & k_1 & \mu \\ -k_1 & 0 & 0 \\ -\mu & 0 & 0 \end{bmatrix} \begin{bmatrix} T \\ N_1 \\ N_2 \end{bmatrix}, \tag{3.9}$$

where μ is a certain function defined by $\mu = g(\Phi T, N_2)$.

Proof. The proof is similar with the proof of Proposition 3.1. □

Proposition 3.4. *Let α be a unit speed N_1-magnetic curve according to Bishop frame in Euclidean 3-space. Then, the curve α is N_1-magnetic trajectory of Killing magnetic vector field V if and only if the Killing magnetic vector field V is*

$$V = -\mu N_1 + k_1 N_2 \tag{3.10}$$

along the curve α.

Proof. The proof is similar with the proof of Proposition 3.2. □

Theorem 3.2. *Let α be a unit speed N_1-magnetic curve according to Bishop frame in a 3-dimensional Euclidean space form M with sectional curvature c and V be a Killing vector field on a simply connected space form $(M(c), g)$. If the curve α is one of the N_1-magnetic trajectories of $(M(c), g, V)$, then we have the following statements:*

i) $\mu = k_2$,

ii) the harmonic curvature function of the curve α according to Bishop frame is

$$H = \frac{k_1''}{k_2''} \tag{3.11}$$

or

$$H = \frac{-k_1 k_1' k_2 - k_2''' - {k_2}^2 k_2' - c k_2' + \frac{k_1 k_1' + k_2 k_2'}{{k_1}^2 + {k_2}^2}(k_2'' + c k_2)}{k_1 k_2 k_2' + k_1''' + {k_1}^2 k_1' + c k_1' - \frac{k_1 k_1' + k_2 k_2'}{{k_1}^2 + {k_2}^2}(k_1'' + c k_1)}. \tag{3.12}$$

Proof. If V is N_1-magnetic vector field in a 3-dimensional Euclidean space form M, then it is the form of (3.10). Differentiating (3.10) with respect to s, from (2.3) we get

$$\nabla_T V = \{\mu k_1 - k_1 k_2\} T - \mu' N_1 + k_1' N_2. \tag{3.13}$$

Using $V(v) = 0$ in Lemma 2.1, we have

$$\mu = k_2. \tag{3.14}$$

Differentiating (3.5) with respect to s and using (3.14), from (2.3) we get

$$\nabla_T^2 V = \{k_1 k_2' - k_1' k_2\} T - k_2'' N_1 + k_1'' N_2 \tag{3.15}$$

and differentiating (3.15) with respect to s once more, from (2.3) we get

$$\nabla_T^3 V = \{2k_1 k_2'' - 2k_1'' k_2\} T + \{{k_1}^2 k_2' - k_1 k_1' k_2 - k_2'''\} N_1 + \{k_1 k_2 k_2' - k_1' {k_2}^2 + k_1'''\} N_2. \tag{3.16}$$

Since $V(\kappa) = 0$ and $V(\tau) = 0$ in Lemma 2.1, using (3.13)-(3.16) and the definition of the Bishop frame, we reach to (3.11) and (3.12), respectively. □

3.3 N_2-Magnetic curves according to Bishop frame in Euclidean 3-space

Definition 3.3. Let $\alpha : I \subset \mathbb{R} \longrightarrow \mathbb{R}^3$ be a curve with Bishop frame in Euclidean 3-space and F_V be a magnetic field in $\mathbb{R}^3$. If the vector field N_2 of the Bishop frame satisfies the Lorentz force equation $\nabla_{\alpha'} N_2 = \Phi(N_2) = V \times N_2$, then the curve α is called an N_2-**magnetic curve according to Bishop frame**.

Proposition 3.5. *Let α be a unit speed N_2-magnetic curve according to Bishop frame in Euclidean 3-space. Then, the Lorentz force according to the Bishop frame is obtained as*

$$\begin{bmatrix} \Phi(T) \\ \Phi(N_1) \\ \Phi(N_2) \end{bmatrix} = \begin{bmatrix} 0 & \gamma & k_2 \\ -\gamma & 0 & 0 \\ -k_2 & 0 & 0 \end{bmatrix} \begin{bmatrix} T \\ N_1 \\ N_2 \end{bmatrix}, \tag{3.17}$$

where γ is a certain function defined by $\gamma = g(\Phi T, N_1)$.

Proof. The proof is similar with the proof of Proposition 3.1. □

Proposition 3.6. *Let α be a unit speed N_2-magnetic curve according to Bishop frame in Euclidean 3-space. Then, the curve α is N_2-magnetic trajectory of Killing magnetic vector field V if and only if the Killing magnetic vector field V is*

$$V = -k_2 N_1 + \gamma N_2 \tag{3.18}$$

along the curve α.

Proof. The proof is similar with the proof of Proposition 3.2. □

Theorem 3.3. *Let α be a unit speed N_2-magnetic curve according to Bishop frame in a 3-dimensional Euclidean space form M with sectional curvature c and V be a Killing vector field on a simply connected space form $(M(c), g)$. If the curve α is one of the N_2-magnetic trajectories of $(M(c), g, V)$, then we have the following statements:*

i) $\gamma = k_1$,

ii) the harmonic curvature function of the curve α according to Bishop frame is

$$H = \frac{k_1''}{k_2''} \tag{3.19}$$

or

$$H = \frac{-k_1 k_1' k_2 - k_2''' - {k_2}^2 k_2' - c k_2' + \frac{k_1 k_1' + k_2 k_2'}{{k_1}^2 + {k_2}^2}(k_2'' + c k_2)}{k_1 k_2 k_2' + k_1''' + {k_1}^2 k_1' + c k_1' - \frac{k_1 k_1' + k_2 k_2'}{{k_1}^2 + {k_2}^2}(k_1'' + c k_1)}. \tag{3.20}$$

Proof. If V is N_2-magnetic vector field in a 3-dimensional Euclidean space form M, then it is the form of (3.18). Differentiating (3.18) with respect to s, from (2.3) we get

$$\nabla_T V = \{k_1 k_2 - \gamma k_2\} T - k_2' N_1 + \gamma' N_2. \tag{3.21}$$

Using $V(v) = 0$ in Lemma 2.1, we have

$$\gamma = k_1 \tag{3.22}$$

and so, the remaining part of the proof is same with the proof of the Theorem 3.2. □

4 Magnetic Curves According to Type-2 Bishop Frame in Euclidean 3-Space

In this section, we investigate the ξ_1-magnetic, ξ_2-magnetic and B-magnetic curves according to type-2 Bishop frame in Euclidean 3-space. Also, we obtain the Killing magnetic vector field V when the curve is a ξ_1-magnetic, ξ_2-magnetic and B-magnetic trajectory of V according to type-2 Bishop frame.

4.1 ξ_1-Magnetic curves according to type-2 Bishop frame in Euclidean 3-space

Definition 4.1. Let $\alpha : I \subset \mathbb{R} \longrightarrow \mathbb{R}^3$ be a curve with type-2 Bishop frame in Euclidean 3-space and F_V be a magnetic field in $\mathbb{R}^3$. If the vector field ξ_1 of the type-2 Bishop frame satisfies the Lorentz force equation $\nabla_{\alpha'}\xi_1 = \Phi(\xi_1) = V \times \xi_1$, then the curve α is called a **ξ_1-magnetic curve according to type-2 Bishop frame**.

Proposition 4.1. *Let α be a unit speed ξ_1-magnetic curve according to type-2 Bishop frame in Euclidean 3-space. Then, the Lorentz force according to the type-2 Bishop frame is obtained as*

$$\begin{bmatrix} \Phi(\xi_1) \\ \Phi(\xi_2) \\ \Phi(B) \end{bmatrix} = \begin{bmatrix} 0 & 0 & -\varepsilon_1 \\ 0 & 0 & \rho_2 \\ \varepsilon_1 & -\rho_2 & 0 \end{bmatrix} \begin{bmatrix} \xi_1 \\ \xi_2 \\ B \end{bmatrix}, \tag{4.1}$$

where ρ_2 is a certain function defined by $\rho_2 = g(\Phi\xi_2, B)$.

Proof. Let α be ξ_1-magnetic curve according to type-2 Bishop frame in Euclidean 3-space with the type-2 Bishop apparatus $\{\xi_1, \xi_2, B, \varepsilon_1, \varepsilon_2\}$. From the definition of the ξ_1-magnetic curve according to type-2 Bishop frame, we know that $\Phi(\xi_1) = -\varepsilon_1 B$. On the other hand, since $\Phi(\xi_2) \in Sp\{\xi_1, \xi_2, B\}$, we have $\Phi(\xi_2) = a_1\xi_1 + a_2\xi_2 + a_3 B$. So, we get

$$\begin{aligned} a_1 &= g(\Phi\xi_2, \xi_1) = -g(\xi_2, \Phi\xi_1) = -g(\xi_2, -\varepsilon_1 B) = 0, \\ a_2 &= g(\Phi\xi_2, \xi_2) = 0, \\ a_3 &= g(\Phi\xi_2, B) = \rho_2 \end{aligned}$$

and hence we obtain that, $\Phi(N) = \rho_2 B$.

Furthermore, from $\Phi(B) = b_1\xi_1 + b_2\xi_2 + b_3 B$, we have

$$\begin{aligned} b_1 &= g(\Phi B, \xi_1) = -g(B, \Phi\xi_1) = -g(B, -\varepsilon_1 B) = \varepsilon_1, \\ b_2 &= g(\Phi B, \xi_2) = -g(B, \Phi\xi_2) = -\rho_2, \\ b_3 &= g(\Phi B, B) = 0 \end{aligned}$$

and so, we can write $\Phi(N_2) = \varepsilon_1\xi_1 - \rho_2\xi_2$, which completes the proof. □

Proposition 4.2. *Let α be a unit speed ξ_1-magnetic curve according to type-2 Bishop frame in Euclidean 3-space. Then, the curve α is ξ_1-magnetic trajectory of Killing magnetic vector field V if and only if the Killing magnetic vector field V is*

$$V = \rho_2\xi_1 + \varepsilon_1\xi_2 \tag{4.2}$$

along the curve α.

Proof. Let α be ξ_1-magnetic trajectory of Killing magnetic vector field V according to type-2 Bishop frame. Using Proposition 4.1 and taking $V = a\xi_1 + b\xi_2 + cB$; from $\Phi(\xi_1) = V \times \xi_1$, we get

$$b = \varepsilon_1, \ \ c = 0;$$

from $\Phi(\xi_2) = V \times \xi_2$, we get

$$a = \rho_2, \ \ c = 0$$

and from $\Phi(B) = V \times B$, we get

$$a = \rho_2, \ \ b = \varepsilon_1$$

and so the Killing magnetic vector field V can be written by (4.2). Conversely, if the Killing magnetic vector field V is the form of (4.2), then one can easily see that $V \times \xi_1 = \Phi(\xi_1)$ holds. So, the curve α is a ξ_1-magnetic projectory of the Killing magnetic vector field V according to type-2 Bishop frame. □

Theorem 4.1. *Let α be a unit speed ξ_1-magnetic curve according to type-2 Bishop frame in a 3-dimensional Euclidean space form M with sectional curvature c and V be a Killing vector field on a simply connected space form $(M(c), g)$. If the curve α is one of the ξ_1-magnetic trajectories of $(M(c), g, V)$, then we have the following equations:*

$$\tan\theta = \frac{\varepsilon_1'}{\rho_2'}, \tag{4.3}$$

$$\tan\theta = \frac{-\rho_2''\theta' + \rho_2{\varepsilon_1}^2\theta' + {\varepsilon_1}^2\varepsilon_2\theta' + 2\rho_2'\varepsilon_1\varepsilon_2 + 2\varepsilon_1'{\varepsilon_2}^2 + \rho_2\varepsilon_1'\varepsilon_2 + \varepsilon_1\varepsilon_2\varepsilon_2' - c\rho_2\theta'}{\varepsilon_1''\theta' - \rho_2\varepsilon_1\varepsilon_2\theta' - \varepsilon_1{\varepsilon_2}^2\theta' + 2\rho_2'{\varepsilon_1}^2 + 2\varepsilon_1'\varepsilon_1\varepsilon_2 + \rho_2\varepsilon_1'\varepsilon_1 + {\varepsilon_1}^2\varepsilon_2' + c\varepsilon_1\theta'} \tag{4.4}$$

and

$$\begin{aligned} & \theta'\{3\rho_2''\varepsilon_1 + 3\rho_2'\varepsilon_1' + 3\varepsilon_1''\varepsilon_2 + 3\varepsilon_1'\varepsilon_2' + \rho_2\varepsilon_1'' + \varepsilon_1\varepsilon_2'' - \rho_2{\varepsilon_1}^3 - \rho_2\varepsilon_1{\varepsilon_2}^2 - {\varepsilon_1}^3\varepsilon_2 - \varepsilon_1{\varepsilon_2}^3\} \\ - \ & \theta''\{2\rho_2'\varepsilon_1 + 2\varepsilon_1'\varepsilon_2 + \rho_2\varepsilon_1' + \varepsilon_1\varepsilon_2'\} + \theta'^3\{\rho_2\varepsilon_1 + \varepsilon_1\varepsilon_2\} + c\theta'\{\rho_2\varepsilon_1 + \varepsilon_1\varepsilon_2\} = 0. \end{aligned} \tag{4.5}$$

Proof. If V is ξ_1-magnetic vector field in a 3-dimensional Euclidean space form M, then it is the form of (4.2). Differentiating (4.2) with respect to s, from (2.5) we get

$$V' = \rho_2'\xi_1 + \varepsilon_1'\xi_2 + \{-\rho_2\varepsilon_1 - \varepsilon_1\varepsilon_2\}B. \tag{4.6}$$

Using $V(v) = 0$ in Lemma 2.1, we have (4.3).

Differentiating (4.6) with respect to s, from (2.5) we get

$$V'' = \{\rho_2'' - \rho_2{\varepsilon_1}^2 - {\varepsilon_1}^2\varepsilon_2\}\xi_1 + \{\varepsilon_1'' - \rho_2\varepsilon_1\varepsilon_2 - \varepsilon_1{\varepsilon_2}^2\}\xi_2 + \{-2\rho_2'\varepsilon_1 - 2\varepsilon_1'\varepsilon_2 - \rho_2\varepsilon_1' - \varepsilon_1\varepsilon_2'\}B. \tag{4.7}$$

Since $V(\kappa) = 0$ in Lemma 2.1, using (4.6) and (4.7), we reach to (4.4).

Differentiating (4.7) with respect to s once more, from (2.5) we get

$$\begin{aligned} V''' \ = \ & \{\rho_2''' - 3\rho_2'{\varepsilon_1}^2 - 3\rho_2\varepsilon_1\varepsilon_1' - 4\varepsilon_1\varepsilon_1'\varepsilon_2 - 2{\varepsilon_1}^2\varepsilon_2'\}\xi_1 \\ & + \{\varepsilon_1''' - 3\rho_2'\varepsilon_1\varepsilon_2 - 2\rho_2\varepsilon_1'\varepsilon_2 - \rho_2\varepsilon_1\varepsilon_2' - 3\varepsilon_1'{\varepsilon_2}^2 - 3\varepsilon_1\varepsilon_2\varepsilon_2'\}\xi_2 \\ & + \{-3\rho_2''\varepsilon_1 - 3\rho_2'\varepsilon_1' - 3\varepsilon_1''\varepsilon_2 - 3\varepsilon_1'\varepsilon_2' - \rho_2\varepsilon_1'' - \varepsilon_1\varepsilon_2'' + \rho_2{\varepsilon_1}^3 \\ & \quad + \rho_2\varepsilon_1{\varepsilon_2}^2 + {\varepsilon_1}^3\varepsilon_2 + \varepsilon_1{\varepsilon_2}^3\}B. \end{aligned} \tag{4.8}$$

Since $V(\tau) = 0$ in Lemma 2.1, using (4.6)-(4.8) and the definition of the type-2 Bishop frame, we reach to (4.5). □

4.2 ξ_2-Magnetic curves according to type-2 Bishop frame in Euclidean 3-space

Definition 4.2. Let $\alpha : I \subset \mathbb{R} \longrightarrow \mathbb{R}^3$ be a curve with type-2 Bishop frame in Euclidean 3-space and F_V be a magnetic field in $\mathbb{R}^3$. If the vector field ξ_2 of the type-2 Bishop frame satisfies the Lorentz force equation $\nabla_{\alpha'}\xi_2 = \Phi(\xi_2) = V \times \xi_2$, then the curve α is called a **ξ_2-magnetic curve according to type-2 Bishop frame**.

Proposition 4.3. *Let α be a unit speed ξ_2-magnetic curve according to type-2 Bishop frame in Euclidean 3-space. Then, the Lorentz force according to the type-2 Bishop frame is obtained as*

$$\begin{bmatrix} \Phi(\xi_1) \\ \Phi(\xi_2) \\ \Phi(B) \end{bmatrix} = \begin{bmatrix} 0 & 0 & \mu_2 \\ 0 & 0 & -\varepsilon_2 \\ -\mu_2 & \varepsilon_2 & 0 \end{bmatrix} \begin{bmatrix} \xi_1 \\ \xi_2 \\ B \end{bmatrix}, \tag{4.9}$$

where μ_2 is a certain function defined by $\mu_2 = g(\Phi\xi_1, B)$.

Proof. The proof is similar with the proof of Proposition 4.1. □

Proposition 4.4. *Let α be a unit speed ξ_2-magnetic curve according to type-2 Bishop frame in Euclidean 3-space. Then, the curve α is ξ_2-magnetic trajectory of Killing magnetic vector field V if and only if the Killing magnetic vector field V is*

$$V = -\varepsilon_2\xi_1 - \mu_2\xi_2 \tag{4.10}$$

along the curve α.

Proof. The proof is similar with the proof of Proposition 4.2. □

Theorem 4.2. *Let α be a unit speed ξ_2-magnetic curve according to type-2 Bishop frame in a 3-dimensional Euclidean space form M with sectional curvature c and V be a Killing vector field on a simply connected space form $(M(c), g)$. If the curve α is one of the ξ_2-magnetic trajectories of $(M(c), g, V)$, then we have the following equations:*

$$\tan\theta = \frac{\mu_2'}{\varepsilon_2'}, \tag{4.11}$$

$$\tan\theta = \frac{\varepsilon_2''\theta' - {\varepsilon_1}^2\varepsilon_2\theta' - \mu_2\varepsilon_1\varepsilon_2\theta' - 2\varepsilon_1\varepsilon_2\varepsilon_2' - 2\mu_2'{\varepsilon_2}^2 - \varepsilon_1'{\varepsilon_2}^2 - \mu_2\varepsilon_2\varepsilon_2' + c\varepsilon_2\theta'}{-\mu_2''\theta' + \varepsilon_1{\varepsilon_2}^2\theta' + \mu_2{\varepsilon_2}^2\theta' - 2{\varepsilon_1}^2\varepsilon_2' - 2\mu_2'\varepsilon_1\varepsilon_2 - \varepsilon_1'\varepsilon_1\varepsilon_2 - \mu_2\varepsilon_1\varepsilon_2' - c\mu_2\theta'} \tag{4.12}$$

and

$$\begin{aligned} &\theta'\{3\varepsilon_1\varepsilon_2'' - {\varepsilon_1}^3\varepsilon_2 - \mu_2{\varepsilon_1}^2\varepsilon_2 + 3\mu_2''\varepsilon_2 - \varepsilon_1{\varepsilon_2}^3 - \mu_2{\varepsilon_2}^3 + 3\varepsilon_1'\varepsilon_2' + 3\mu_2'\varepsilon_2' + \varepsilon_1''\varepsilon_2 + \mu_2\varepsilon_2''\} \\ - \; &\theta''\{2\varepsilon_1\varepsilon_2' + 2\mu_2'\varepsilon_2 + \varepsilon_1'\varepsilon_2 + \mu_2\varepsilon_2'\} + \theta'^3\{\varepsilon_1\varepsilon_2 + \mu_2\varepsilon_2\} + c\theta'\{\varepsilon_1\varepsilon_2 + \mu_2\varepsilon_2\} = 0. \end{aligned} \tag{4.13}$$

Proof. If V is ξ_2-magnetic vector field in a 3-dimensional Euclidean space form M, then it is the form of (4.10). Differentiating (4.10) with respect to s, from (2.5) we get

$$V' = -\varepsilon_2'\xi_1 - \mu_2'\xi_2 + \{\varepsilon_1\varepsilon_2 + \mu_2\varepsilon_2\}B. \tag{4.14}$$

Using $V(v) = 0$ in Lemma 2.1, we have (4.11).

Differentiating (4.14) with respect to s, from (2.5) we get

$$V'' = \{-\varepsilon_2'' + {\varepsilon_1}^2\varepsilon_2 + \mu_2\varepsilon_1\varepsilon_2\}\xi_1 + \{-\mu_2'' + \varepsilon_1{\varepsilon_2}^2 + \mu_2{\varepsilon_2}^2\}\xi_2 + \{2\varepsilon_1\varepsilon_2' + 2\mu_2'\varepsilon_2 + \varepsilon_1'\varepsilon_2 + \mu_2\varepsilon_2'\}B. \tag{4.15}$$

Since $V(\kappa) = 0$ in Lemma 2.1, using (4.14) and (4.15), we reach to (4.12).

Differentiating (4.15) with respect to s once more, from (2.5) we get

$$\begin{aligned} V''' &= \{-\varepsilon_2''' + 3\varepsilon_1\varepsilon_1'\varepsilon_2 + 3{\varepsilon_1}^2\varepsilon_2' + 3\mu_2'\varepsilon_1\varepsilon_2 + 2\mu_2\varepsilon_1\varepsilon_2' + \mu_2\varepsilon_1'\varepsilon_2\}\xi_1 \\ &\quad +\{-\mu_2''' + 2\varepsilon_1'{\varepsilon_2}^2 + 4\varepsilon_1\varepsilon_2\varepsilon_2' + 3\mu_2'{\varepsilon_2}^2 + 3\mu_2\varepsilon_2\varepsilon_2'\}\xi_2 \\ &\quad +\{3\varepsilon_1\varepsilon_2'' - {\varepsilon_1}^3\varepsilon_2 - \mu_2{\varepsilon_1}^2\varepsilon_2 + 3\mu_2''\varepsilon_2 - \varepsilon_1{\varepsilon_2}^3 - \mu_2{\varepsilon_2}^3 \\ &\qquad + 3\varepsilon_1'\varepsilon_2' + 3\mu_2'\varepsilon_2' + \varepsilon_1''\varepsilon_2 + \mu_2\varepsilon_2''\}B. \end{aligned} \tag{4.16}$$

Since $V(\tau) = 0$ in Lemma 2.1, using (4.14)-(4.16) and the definition of the type-2 Bishop frame, we reach to (4.13). □

4.3 B-Magnetic curves according to type-2 Bishop frame in Euclidean 3-space

Definition 4.3. Let $\alpha : I \subset \mathbb{R} \longrightarrow \mathbb{R}^3$ be a curve with type-2 Bishop frame in Euclidean 3-space and F_V be a magnetic field in $\mathbb{R}^3$. If the vector field B of the type-2 Bishop frame satisfies the Lorentz force equation $\nabla_{\alpha'} B = \Phi(B) = V \times B$, then the curve α is called a **B-magnetic curve according to type-2 Bishop frame**.

Proposition 4.5. *Let α be a unit speed B-magnetic curve according to type-2 Bishop frame in Euclidean 3-space. Then, the Lorentz force according to the type-2 Bishop frame is obtained as*

$$\begin{bmatrix} \Phi(\xi_1) \\ \Phi(\xi_2) \\ \Phi(B) \end{bmatrix} = \begin{bmatrix} 0 & \gamma_2 & -\varepsilon_1 \\ -\gamma_2 & 0 & -\varepsilon_2 \\ \varepsilon_1 & \varepsilon_2 & 0 \end{bmatrix} \begin{bmatrix} \xi_1 \\ \xi_2 \\ B \end{bmatrix}, \tag{4.17}$$

where γ_2 is a certain function defined by $\gamma_2 = g(\Phi\xi_1, \xi_2)$.

Proof. The proof is similar with the proof of Proposition 4.1. □

Proposition 4.6. *Let α be a unit speed B-magnetic curve according to type-2 Bishop frame in Euclidean 3-space. Then, the curve α is B-magnetic trajectory of Killing magnetic vector field V if and only if the Killing magnetic vector field V is*

$$V = -\varepsilon_2\xi_1 + \varepsilon_1\xi_2 + \gamma_2 B \tag{4.18}$$

along the curve α.

Proof. The proof is similar with the proof of Proposition 4.2. □

Theorem 4.3. *Let α be a unit speed B-magnetic curve according to type-2 Bishop frame in a 3-dimensional Euclidean space form M with sectional curvature c and V be a Killing vector field on a simply connected space form $(M(c), g)$. If the curve α is one of the B-magnetic trajectories of $(M(c), g, V)$, then we have the following equations:*

$$\tan\theta = \frac{\varepsilon_1' + \gamma_2\varepsilon_2}{-\varepsilon_2' + \gamma_2\varepsilon_1}, \tag{4.19}$$

$$\tan\theta = \frac{\begin{array}{c}\{\varepsilon_2''\theta' - 2\gamma_2'\varepsilon_1\theta' - \gamma_2\varepsilon_1'\theta' - \varepsilon_1\varepsilon_2\varepsilon_2' + \varepsilon_1'{\varepsilon_2}^2 + \gamma_2{\varepsilon_1}^2\varepsilon_2 + \gamma_2{\varepsilon_2}^3 \\ -\gamma_2'''\varepsilon_2 + 2\theta'^2\varepsilon_1' + 2\theta'^2\gamma_2\varepsilon_2 + c\varepsilon_2\theta' - c\gamma_2\varepsilon_2\}\end{array}}{\begin{array}{c}\{\varepsilon_1''\theta' + 2\gamma_2'\varepsilon_2\theta' + \gamma_2\varepsilon_2'\theta' - {\varepsilon_1}^2\varepsilon_2' + \varepsilon_1\varepsilon_1'\varepsilon_2 + \gamma_2{\varepsilon_1}^3 + \gamma_2\varepsilon_1{\varepsilon_2}^2 \\ -\gamma_2'''\varepsilon_1 - 2\theta'^2\varepsilon_2' + 2\theta'^2\gamma_2\varepsilon_1 + c\varepsilon_1\theta' - c\gamma_2\varepsilon_1\}\end{array}} \tag{4.20}$$

and

$$\begin{aligned} &\theta'\{2\varepsilon_1\varepsilon_2'' - 2\varepsilon_1''\varepsilon_2 - 3\gamma_2'{\varepsilon_1}^2 - 3\gamma_2'{\varepsilon_2}^2 - 3\gamma_2\varepsilon_1\varepsilon_1' - 3\gamma_2\varepsilon_2\varepsilon_2' + \gamma_2'''\} \\ -\ &\theta''\{\varepsilon_1\varepsilon_2' - \varepsilon_1'\varepsilon_2 - \gamma_2{\varepsilon_1}^2 - \gamma_2{\varepsilon_2}^2 + \gamma_2''\} + \theta'^3\gamma_2' + c\theta'\gamma_2' - c\theta''\gamma_2 = 0. \end{aligned} \tag{4.21}$$

Proof. If V is B-magnetic vector field in a 3-dimensional Euclidean space form M, then it is the form of (4.18). Differentiating (4.18) with respect to s, from (2.5) we get

$$V' = \{-\varepsilon_2' + \gamma_2\varepsilon_1\}\xi_1 + \{\varepsilon_1' + \gamma_2\varepsilon_2\}\xi_2 + \gamma_2' B. \tag{4.22}$$

Using $V(v) = 0$ in Lemma 2.1, we have (4.19).

Differentiating (4.22) with respect to s, from (2.5) we get

$$V'' = \{-\varepsilon_2'' + 2\gamma_2'\varepsilon_1 + \gamma_2\varepsilon_1'\}\xi_1 + \{\varepsilon_1'' + 2\gamma_2'\varepsilon_2 + \gamma_2\varepsilon_2'\}\xi_2 + \{\varepsilon_1\varepsilon_2' - \varepsilon_1'\varepsilon_2 - \gamma_2{\varepsilon_1}^2 - \gamma_2{\varepsilon_2}^2 + \gamma_2''\}B. \tag{4.23}$$

Since $V(\kappa) = 0$ in Lemma 2.1, using (4.22) and (4.23), we reach to (4.20).

Differentiating (4.23) with respect to s once more, from (2.5) we get

$$\begin{aligned} V''' &= \{-\varepsilon_2''' + 2\gamma_2''\varepsilon_1 + 3\gamma_2'\varepsilon_1' + \gamma_2\varepsilon_1'' + {\varepsilon_1}^2\varepsilon_2' - \varepsilon_1\varepsilon_1'\varepsilon_2 - \gamma_2{\varepsilon_1}^3 - \gamma_2\varepsilon_1{\varepsilon_2}^2 + \gamma_2''\varepsilon_1\}\xi_1 \\ &\quad + \{\varepsilon_1''' + 2\gamma_2''\varepsilon_2 + 3\gamma_2'\varepsilon_2' + \gamma_2\varepsilon_2'' + \varepsilon_1\varepsilon_2\varepsilon_2' - \varepsilon_1'{\varepsilon_2}^2 - \gamma_2{\varepsilon_1}^2\varepsilon_2 - \gamma_2{\varepsilon_2}^3 + \gamma_2''\varepsilon_2\}\xi_2 \\ &\quad + \{2\varepsilon_1\varepsilon_2'' - 2\varepsilon_1''\varepsilon_2 - 3\gamma_2'{\varepsilon_1}^2 - 3\gamma_2'{\varepsilon_2}^2 - 3\gamma_2\varepsilon_1\varepsilon_1' - 3\gamma_2\varepsilon_2\varepsilon_2' + \gamma_2'''\}B. \end{aligned} \tag{4.24}$$

Since $V(\tau) = 0$ in Lemma 2.1, using (4.22)-(4.24) and the definition of the type-2 Bishop frame, we reach to (4.21). □

Now, we will give an example for magnetic curves according to Bishop frame and type-2 Bishop frame.

Example 4.4. *Let us consider the curve*

$$\alpha(t) = \left(\cos\frac{t}{\sqrt{2}}, \sin\frac{t}{\sqrt{2}}, \frac{t}{\sqrt{2}}\right), \tag{4.25}$$

which is a unit speed circular helix in E^3. Here, one can easily calculate its Frenet-Serret trihedra and curvatures as

$$\begin{aligned} T &= \frac{1}{\sqrt{2}}\left(-\sin\frac{t}{\sqrt{2}}, \cos\frac{t}{\sqrt{2}}, 1\right), \\ N &= \left(-\cos\frac{t}{\sqrt{2}}, -\sin\frac{t}{\sqrt{2}}, 0\right), \\ B &= \frac{1}{\sqrt{2}}\left(\sin\frac{t}{\sqrt{2}}, -\cos\frac{t}{\sqrt{2}}, 1\right), \\ \kappa &= \tau = \frac{1}{2}, \end{aligned} \tag{4.26}$$

respectively. Now, we will obtain the Bishop apparatus of the curve α. For this, we find the $\theta(t)$ with the aid of $\tau(t) = \theta'(t)$ as

$$\theta(t) = \int_0^t \frac{1}{2}dt = \frac{t}{2}. \tag{4.27}$$

So, the transformation matrix can be expressed as

$$\begin{bmatrix} T \\ N \\ B \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\frac{t}{2} & \sin\frac{t}{2} \\ 0 & -\sin\frac{t}{2} & \cos\frac{t}{2} \end{bmatrix} \begin{bmatrix} T \\ N_1 \\ N_2 \end{bmatrix}. \tag{4.28}$$

Using the method of Cramer, we can obtain the Bishop trihedra of the curve α as follows

$$T = \frac{1}{\sqrt{2}}\left(-\sin\frac{t}{\sqrt{2}}, \cos\frac{t}{\sqrt{2}}, 1\right), \tag{4.29}$$

$$N_1 = \left(-\frac{1}{\sqrt{2}}\sin\frac{t}{\sqrt{2}}\sin\frac{t}{2} - \cos\frac{t}{\sqrt{2}}\cos\frac{t}{2}, \frac{1}{\sqrt{2}}\cos\frac{t}{\sqrt{2}}\sin\frac{t}{2} - \sin\frac{t}{\sqrt{2}}\cos\frac{t}{2}, -\frac{1}{\sqrt{2}}\sin\frac{t}{2}\right),$$

$$N_2 = \left(\frac{1}{\sqrt{2}}\sin\frac{t}{\sqrt{2}}\cos\frac{t}{2} - \cos\frac{t}{\sqrt{2}}\sin\frac{t}{2}, -\frac{1}{\sqrt{2}}\cos\frac{t}{\sqrt{2}}\cos\frac{t}{2} - \sin\frac{t}{\sqrt{2}}\sin\frac{t}{2}, \frac{1}{\sqrt{2}}\cos\frac{t}{2}\right)$$

and the Bishop curvatures can be obtain as

$$
\begin{aligned}
k_1 &= \frac{1}{2}\cos\frac{t}{2}, \\
k_2 &= \frac{1}{2}\sin\frac{t}{2}.
\end{aligned} \tag{4.30}
$$

Now, let us find the magnetic vector field V when the curve (4.25) is a T-magnetic, N_1-magnetic or N_2-magnetic trajectory of the magnetic vector field V according to Bishop frame (4.29):

i) *If the curve α is T-magnetic according to Bishop frame, then from Theorem 3.1 we have $\rho =$constant. Taking $\rho = c =$constant and using (4.29) and (4.30) in (3.2), the magnetic vector field V is obtained as*

$$V = \frac{1}{\sqrt{2}}\left(\sin\frac{t}{\sqrt{2}}(-c+\frac{1}{2}), \cos\frac{t}{\sqrt{2}}(c-\frac{1}{2}), c+\frac{1}{2}\right). \tag{4.31}$$

Here, it can be seen that, from (3.1), (4.29) and (4.31),

$$\nabla_{\alpha'}T = \Phi(T) = V \times T = \frac{1}{2}\left(-\cos\frac{t}{\sqrt{2}}, -\sin\frac{t}{\sqrt{2}}, 0\right)$$

satisfies. So, the curve α is a T-magnetic curve according to Bishop frame with the magnetic vector field (4.31).

ii) *If the curve α is N_1-magnetic according to Bishop frame, then from Theorem 3.2 we have $\mu = k_2$. Taking $\mu = \frac{1}{2}\sin\frac{t}{2}$ and using (4.29) and (4.30) in (3.10), we obtain the magnetic vector field V as*

$$V = \frac{1}{2\sqrt{2}}\left(\sin\frac{t}{\sqrt{2}}, -\cos\frac{t}{\sqrt{2}}, 1\right). \tag{4.32}$$

Here, it can be seen that, from (3.9), (4.29) and (4.32),

$$\nabla_{\alpha'}N_1 = \Phi(N_1) = V \times N_1 = \frac{1}{2\sqrt{2}}\cos\frac{t}{2}\left(\sin\frac{t}{\sqrt{2}}, -\cos\frac{t}{\sqrt{2}}, -1\right)$$

satisfies. So, the curve α is an N_1-magnetic curve according to Bishop frame with the magnetic vector field (4.32).

iii) *If the curve α is N_2-magnetic according to Bishop frame, then from Theorem 3.3 we have $\gamma = k_1$. Taking $\gamma = \frac{1}{2}\cos\frac{t}{2}$ and using (4.29) and (4.30) in (3.18), we obtain the magnetic vector field V as*

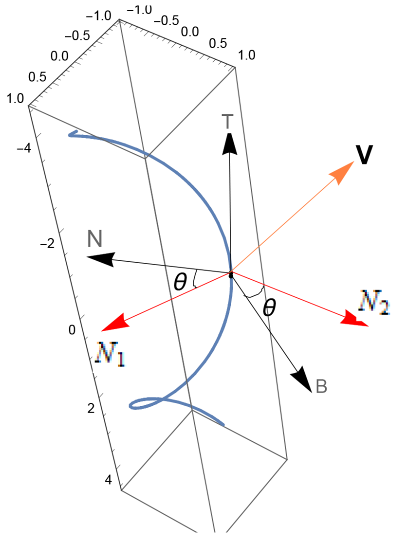
$$V = \frac{1}{2\sqrt{2}}\left(\sin\frac{t}{\sqrt{2}}, -\cos\frac{t}{\sqrt{2}}, 1\right). \tag{4.33}$$

Here, it can be seen that, from (3.17), (4.29) and (4.33),

$$\nabla_{\alpha'}N_2 = \Phi(N_2) = V \times N_2 = \frac{1}{2\sqrt{2}}\sin\frac{t}{2}\left(\sin\frac{t}{\sqrt{2}}, -\cos\frac{t}{\sqrt{2}}, -1\right)$$

satisfies. So, the curve α is an N_2-magnetic curve according to Bishop frame with the magnetic vector field (4.33).

When the curve α is T-magnetic according to Bishop frame, the figure of α and V can be drawn as following:



Similarly, if the curve α is N_1-magnetic and N_2-magnetic according to Bishop frame, one can draw the figure of α and V as above.

Now, we will obtain the type-2 Bishop apparatus of the curve (4.25). From $\kappa(t) = \theta'(t)$, we find

$$\theta(t) = \int_0^t \frac{1}{2} dt = \frac{t}{2}. \tag{4.34}$$

So, the transformation matrix can be expressed as

$$\begin{bmatrix} T \\ N \\ B \end{bmatrix} = \begin{bmatrix} \sin\frac{t}{2} & -\cos\frac{t}{2} & 0 \\ \cos\frac{t}{2} & \sin\frac{t}{2} & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \xi_1 \\ \xi_2 \\ B \end{bmatrix}. \tag{4.35}$$

Using the method of Cramer, we can obtain the type-2 Bishop trihedra of the curve α as follows

$$\begin{aligned} \xi_1 &= \left(-\frac{1}{\sqrt{2}}\sin\frac{t}{\sqrt{2}}\sin\frac{t}{2} - \cos\frac{t}{\sqrt{2}}\cos\frac{t}{2}, \frac{1}{\sqrt{2}}\cos\frac{t}{\sqrt{2}}\sin\frac{t}{2} - \sin\frac{t}{\sqrt{2}}\cos\frac{t}{2}, \frac{1}{\sqrt{2}}\sin\frac{t}{2}\right), \\ \xi_2 &= \left(\frac{1}{\sqrt{2}}\sin\frac{t}{\sqrt{2}}\cos\frac{t}{2} - \cos\frac{t}{\sqrt{2}}\sin\frac{t}{2}, -\frac{1}{\sqrt{2}}\cos\frac{t}{\sqrt{2}}\cos\frac{t}{2} - \sin\frac{t}{\sqrt{2}}\sin\frac{t}{2}, -\frac{1}{\sqrt{2}}\cos\frac{t}{2}\right), \\ B &= \frac{1}{\sqrt{2}}\left(\sin\frac{t}{\sqrt{2}}, -\cos\frac{t}{\sqrt{2}}, 1\right) \end{aligned} \tag{4.36}$$

and the type-2 Bishop curvatures can be obtain as

$$\begin{aligned} \varepsilon_1 &= -\frac{1}{2}\cos\frac{t}{2}, \\ \varepsilon_2 &= -\frac{1}{2}\sin\frac{t}{2}. \end{aligned} \tag{4.37}$$

Now, let us find the magnetic vector field V when the curve (4.25) is a ξ_1-magnetic, ξ_2-magnetic or B-magnetic trajectory of the magnetic vector field V according to type-2 Bishop frame (4.36):

i°) *If the curve α is ξ_1-magnetic according to type-2 Bishop frame, then from Theorem 4.1 we have $\tan\theta = \frac{\varepsilon_1'}{\rho_2'}$. So, we have $\rho_2 = \frac{1}{2}\sin\frac{t}{2} + k$, $k =$constant. Using (4.36) and (4.37) in (4.2), we obtain the magnetic vector field V as*

$$V = \frac{1}{2\sqrt{2}}\left(-\sin\frac{t}{\sqrt{2}}, \cos\frac{t}{\sqrt{2}}, 1\right) + k.\xi_1. \tag{4.38}$$

Here, it can be seen that, from (4.1), (4.36) and (4.38),

$$\nabla_{\alpha'}\xi_1 = \Phi(\xi_1) = V \times \xi_1 = \frac{1}{2\sqrt{2}}\cos\frac{t}{2}\left(\sin\frac{t}{\sqrt{2}}, -\cos\frac{t}{\sqrt{2}}, 1\right)$$

satisfies. So, the curve α *is a* ξ_1*-magnetic curve according to type-2 Bishop frame with the magnetic vector field (4.38).*

ii°) *If the curve* α *is* ξ_2*-magnetic according to type-2 Bishop frame, then from Theorem 4.2 we have* $\tan\theta = \frac{\mu_2'}{\varepsilon_2'}$. *So, we have* $\mu_2 = \frac{1}{2}\cos\frac{t}{2} + l$, $l =$*constant. Using (4.36) and (4.37) in (4.10), we obtain the magnetic vector field* V *as*

$$V = \frac{1}{2\sqrt{2}}\left(-\sin\frac{t}{\sqrt{2}}, \cos\frac{t}{\sqrt{2}}, 1\right) - l.\xi_2. \tag{4.39}$$

Here, it can be seen that, from (4.9), (4.36) and (4.39),

$$\nabla_{\alpha'}\xi_2 = \Phi(\xi_2) = V \times \xi_2 = \frac{1}{2\sqrt{2}}\sin\frac{t}{2}\left(\sin\frac{t}{\sqrt{2}}, -\cos\frac{t}{\sqrt{2}}, 1\right)$$

satisfies. So, the curve α *is a* ξ_2*-magnetic curve according to type-2 Bishop frame with the magnetic vector field (4.39).*

iii°) *If the curve* α *is* B*-magnetic according to type-2 Bishop frame, then from Theorem 4.3 we have* $\tan\theta = \frac{\varepsilon_1' + \gamma_2\varepsilon_2}{-\varepsilon_2' + \gamma_2\varepsilon_1}$. *So, the last equation satisfies for* $\gamma_2 = 0$. *Taking* $\gamma_2 = 0$ *and using (4.36) and (4.37) in (4.18), we obtain the magnetic vector field* V *as*

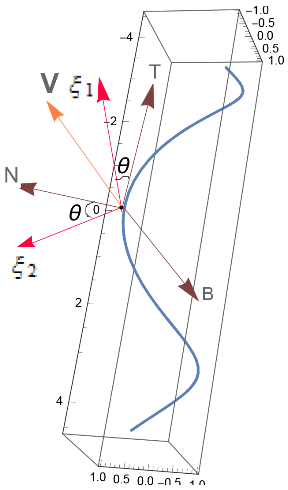
$$V = \frac{1}{2\sqrt{2}}\left(-\sin\frac{t}{\sqrt{2}}, \cos\frac{t}{\sqrt{2}}, 1\right). \tag{4.40}$$

Here, it can be seen that, from (4.17), (4.36) and (4.40),

$$\nabla_{\alpha'}B = \Phi(B) = V \times B = \frac{1}{2}\left(\cos\frac{t}{\sqrt{2}}, \sin\frac{t}{\sqrt{2}}, 0\right)$$

satisfies. So, the curve α *is a* B*-magnetic curve according to type-2 Bishop frame with the magnetic vector field (4.40).*

When the curve α *is* ξ_1*-magnetic according to type-2 Bishop frame, the figure of* α *and* V *can be drawn as following:*



Remark 4.1. Taking $k = 0$ and $l = 0$ in (4.38) and (4.39), respectively, one can see that the magnetic vector fields (4.38), (4.39) and (4.40) are equal when the curve (4.25) is ξ_1-magnetic, ξ_2-magnetic and B-magnetic curve according to type-2 Bishop frame.

5 Conclusions

In the present study, we construct the notion of different types of magnetic curves according to Bishop frame and type-2 Bishop frame in Euclidean 3-space and give some examples for them. The results which are presented in this paper are interesting and important in this field. We hope that, this study can bring a new viewpoint for geometers who want to study about magnetic curves in other spaces.

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Competing Interests

Authors have declared that no competing interests exist.

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